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# LOW REYNOLDS NUMBER MOTION OF TWO DROPS SUBMERGED IN AN UNBOUNDED ARBITRARY VELOCITY FIELD

G. HETSRONI† and S. HABER
Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

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**Abstract**—The hydrodynamics of two droplets submerged in an unbounded arbitrary velocity field is studied by solving Stokes' equations for the flow fields in and around the droplets by means of the reflection method. Solutions are obtained for the drag forces and the terminal settling velocities of two droplets moving in an unbounded quiescent fluid in a gravitational field.

## 1. INTRODUCTION

The motion of two droplets submerged in an unbounded arbitrary velocity field is yet an unsolved problem. Its significance lies in the fact that it is a necessary first step in solutions relevant to emulsions.

In calculating the phenomenological properties of dilute emulsions, one is mainly concerned with two properties: the velocity of the droplets in the emulsions and the rheological equation of the emulsion. The former is applicable to the equation of conservation of the volumetric concentration, while the latter is applicable to the momentum equation (Batchelor 1972; Batchelor & Green 1972*b*).

A second problem of interest is that of meteorological and cloud physics. It is well established that cloud formation starts when water vapor condenses on micrometer size particles to form small droplets, typically a few micrometers in diameter. These droplets then coalesce to form drops a few hundred micrometers in diameter, which then proceed to grow, eventually falling as rain. It is accepted that the principle mechanism of this coalescence is that of collision. Therefore, one of the major problems is the computation of collision efficiencies. These collision efficiencies can be obtained only when the problem of the motion of two unequal drops in a quiescent unbounded velocity field is solved.

There exists substantial literature devoted mainly to *rigid* spheres moving in various configurations, e.g. the motion of two rigid spheres along their line of centers (Stimson & Jeffery 1926), the motion of a sphere towards a wall (Brenner 1961) and lately the solution for two equal rigid spheres moving perpendicular to their line of centers (Goldman, Cox & Brenner 1966) and for two unequal spheres in the same configuration (O'Neill & Majumdar 1970). Approximate solutions were obtained for the case of two rigid spheres suspended in shear flow (Lin, Lee & Sather 1970; Batchelor & Green 1972*a*; Brenner & O'Neill 1972; and for touching spheres by Nir & Acrivos 1973).

As for drops, an exact solution was obtained for two unequal drops moving along their line of centers (Haber *et al.* 1974) which included the effect of different viscosities in the drips and the solution for two touching drops.

The solution for two drops moving perpendicular to their line of centers has not previously been obtained. An approximate solution is presented herein which uses the method of reflection to calculate the flow fields in and around the droplets, the drag forces on the droplets, and their terminal settling velocities.

The method of reflection is well described by Happel & Brenner (1965) for the solution of two *rigid* spheres moving arbitrarily in an unbounded medium.

†Present address: EPRI. Box 10412, Palo Alto, CA94303, U.S.A.

## 2. STATEMENT OF THE PROBLEM

The problem considered herein is that of two liquid droplets moving arbitrarily in an unbounded medium.

The fluids involved are assumed to be homogeneous, isothermal, Newtonian and of constant densities. The unperturbed flow field $\mathbf{v}_\infty$ is Stokesian, but other than that is quite arbitrary. Surface active agents are absent from the system.

Two spherical coordinate systems are used, i.e. $R$, $\Theta$, $\Phi$, and $r$, $\theta$, $\phi$, whose origins coincide with the center of the droplets '$a$' and '$b$' (figure 1). The droplets are initially at a distance $l$ apart. The motion of the droplet is such that the two coordinate systems can be assumed to be inertial, and the flow field is in quasi-steady-state.

It is further assumed that the droplets are small and move with low relative velocity, such that the inertia terms in the equations of motion can be neglected.

With these suppositions the equations of motion and continuity are as follows:

$$\mu_e \nabla^2 \mathbf{v} = \nabla p_e, \tag{1a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1b}$$

where the subscript $e$ indicates a property exterior to the droplet.

For the flow interior to droplet '$a$'

$$\mu_a \nabla^2 \mathbf{u} = \nabla p_i, \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2b}$$

For the interior of droplet '$b$',

$$\mu_b \nabla^2 \mathbf{U} = \nabla P_i, \tag{3a}$$

$$\nabla \cdot \mathbf{U} = 0. \tag{3b}$$

Equations [1] to [3] are to be solved subject to the following boundary conditions:

Far from the droplets the flow field is unperturbed, i.e. at $R = \infty$ and at $r=\infty$

$$\mathbf{v}=\mathbf{v}_\infty. \tag{4}$$

On the interface of the droplets it is assumed that the velocities are continuous and the normal stress vary by a term proportional to the surface tension, viz. at $r = a$

$$\mathbf{v}^* = \mathbf{u}^*, \tag{5a}$$

$$\mathbf{u}^* \cdot \mathbf{t}_r = \mathbf{u}_a \cdot \mathbf{t}_r, \tag{5b}$$

$$\boldsymbol{\pi}^*_{(r)} = \boldsymbol{\tau}^*_{(r)} + \sigma_a\left(\frac{1}{R_{a1}} + \frac{1}{R_{a2}}\right)\mathbf{t}_r, \tag{5c}$$

at $R = b$

$$\mathbf{v}^{**} = \mathbf{U}^{**}, \tag{6a}$$

$$\mathbf{U}^{**} \cdot \mathbf{t}_R = \mathbf{U}_b \cdot \mathbf{t}_R, \tag{6b}$$

$$\boldsymbol{\pi}^{**}_{(R)} = \mathbf{T}^{**}_{(R)} + \sigma_b\left(\frac{1}{R_{b1}} + \frac{1}{R_{b2}}\right)\mathbf{t}_R, \tag{6c}$$

where an asterisk (*) indicates that the functions are to be evaluated at the interface $r = a$, and the double asterisk (**) indicate that the functions are to be evaluated at the interface $R = b$; $\mathbf{v}$, $\mathbf{u}$, $\mathbf{U}$, $p_e$, $p_i$, $P_i$, are the velocity vectors and pressures exterior to the droplet and interior to droplet '$a$' and '$b$', respectively; $\boldsymbol{\pi}_{(r)}$, $\boldsymbol{\tau}_{(r)}$, and $\mathbf{T}_{(R)}$ are the normal stress vectors based on the velocities exterior to the droplets and interior to droplet '$a$' and '$b$', respectively; $\sigma_a$ and $\sigma_b$ are the respective surface tensions, while $R_{a1}$, $R_{a2}$, and $R_{b1}$, $R_{b2}$ are the principal radii of the two droplets; $\mathbf{t}_r$ and $\mathbf{t}_R$ are unit vectors normal to the interface of droplet '$a$' and '$b$', respectively.

## 3. THE SOLUTION

The solution of [1] to [3], subject to the boundary conditions [4] to [6] should yield, simultaneously, the flow fields in and around the droplets, and the geometry of their interfaces. It was shown previously (Hetsroni & Haber 1970; Hetsroni *et al.* 1970) that if the geometry of the interface is assumed *a priori*, an inconsistency may result. On the other hand, the simultaneous determination of the velocity fields and geometry is exceedingly difficult and is not attempted here. Instead, an iterative procedure is adopted, similar to our previous works (Hetsroni & Haber 1970; Hetsroni *et al.* 1970; Haber & Hetsroni 1971). The solution is thus initiated by assuming that the droplets are spherical and solving the flow fields. Subsequently, the geometry of the interface is solved for these flow fields. This new interface can be used for solving new flow fields and the procedure can be continued until the desired accuracy is reached. Here we perform only the first iteration, i.e. for spherical droplets, and the solution is applicable only for

$$\mathfrak{D}_i \equiv \frac{\mu_e|\mathbf{v}_\infty|}{\sigma_i} \ll 1, \qquad (i = a, b)$$

where $\mu_e$ is the viscosity of the continuous field, $\mathbf{v}_\infty$ is a velocity scale, and $\sigma_i$ is the surface tension of droplet '$a$' or droplet '$b$'. Therefore, we shall utilize only twelve out of the fourteen available boundary conditions. The remaining boundary conditions can then be used for calculating the geometry of the surface.

Since the Stokes' equations of motion are linear, the following definitions are permissible, and are convenient:

$$\left.\begin{array}{lll} \mathbf{v} = \mathbf{v}_\infty + \mathbf{v}_1 + \mathbf{v}_2; & p_e = p_\infty + p_1 + p_2; & \boldsymbol{\pi}_{(r)} = \boldsymbol{\pi}_{(r)\infty} + \boldsymbol{\pi}_{(r)1} + \boldsymbol{\pi}_{(r)2} \\ \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2; & p_i = p_{i1} + p_{i2}; & \boldsymbol{\tau}_{(r)} = \boldsymbol{\tau}_{(r)1} + \boldsymbol{\tau}_{(r)2} \\ \mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2; & P_i = P_{i1} + P_{i2}; & \mathbf{T}_{(R)} = \mathbf{T}_{(R)1} + \mathbf{T}_{(R)2}. \end{array}\right\} \qquad [7]$$

With these expressions, the boundary conditions [5] and [6] are rewritten as follows: at $r = \infty$

$$\mathbf{v}_1 = 0, \qquad [8a]$$

at $r = a$

$$\mathbf{v}_\infty^* + \mathbf{v}_1^* = \mathbf{u}_1^*, \qquad [8b]$$

$$\mathbf{u}_1^* \cdot \mathbf{t}_r = \mathbf{U}_a \cdot \mathbf{t}_r, \qquad [8c]$$

$$(\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot (\overset{*}{\boldsymbol{\pi}}_{(r)\infty} + \overset{*}{\boldsymbol{\pi}}_{(r)1}) = (\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot \overset{*}{\boldsymbol{\tau}}_{(r)1}, \qquad [8d]$$

at $R = b$

$$\mathbf{v}_1^{**} = \mathbf{U}_1^{**}, \qquad [8e]$$

$$\mathbf{v}_1^{**} \cdot \mathbf{t}_R = 0, \qquad [8f]$$

$$(\mathbf{I} - \mathbf{t}_R\mathbf{t}_R) \cdot \overset{**}{\boldsymbol{\pi}}_{(R)1} = (\mathbf{I} - \mathbf{t}_R\mathbf{t}_R) \cdot \overset{**}{\mathbf{T}}_{(R)1}, \qquad [8g]$$

for $\mathbf{v}_2$, $\mathbf{u}_2$, and $\mathbf{U}_2$, at $r = \infty$

$$\mathbf{v}_2 = 0, \qquad [9a]$$

at $r = a$

$$\mathbf{v}_2^* = \mathbf{u}_2^*, \qquad [9b]$$

$$\mathbf{u}_2^* \cdot \mathbf{t}_r = 0, \qquad [9c]$$

$$(\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot \boldsymbol{\pi}^*_{(r)2} = (\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot \boldsymbol{\tau}^*_{(r)2}, \qquad [9d]$$

at $R = b$

$$\mathbf{v}_\infty^{**} + \mathbf{v}_2^{**} = \mathbf{U}_2^{**}, \qquad [9e]$$

$$\mathbf{U}_2^{**} \cdot \mathbf{t}_R = \mathbf{U}_b \cdot \mathbf{t}_R, \qquad [9f]$$

$$(\mathbf{I} - \mathbf{t}_R\mathbf{t}_R) \cdot (\boldsymbol{\pi}^{**}_{(R)\infty} + \boldsymbol{\pi}^{**}_{(R)2}) = (\mathbf{I} - \mathbf{t}_R\mathbf{t}_R) \cdot \mathbf{T}^{**}_{(R)2}. \qquad [9g]$$

It is obvious from these boundary conditions that the solutions for $\mathbf{v}_1$, $\mathbf{u}_1$, and $\mathbf{U}_1$ are identical to the solutions of $\mathbf{v}_2$, $\mathbf{u}_2$, and $\mathbf{U}_2$, respectively. We therefore limit ourselves to the solutions of $\mathbf{v}_1$, $\mathbf{u}_1$, and $\mathbf{U}_1$.

The solution is based on the method of reflection, which is described elsewhere (Happel & Brenner 1965; Hetsroni & Haber 1970). It consists of a sum of velocity fields, all of which satisfy [1] for the velocity field of the continuous medium, [2] for the velocity field interior to droplet '$a$', [3] for the interior of droplet '$b$'. Each of the solutions partially satisfies the boundary conditions.

The reflected fields are

$$\mathbf{v}_1 = \sum_{i=1}^{\infty} \mathbf{v}_{1i}, \qquad [10a]$$

$$\mathbf{u}_1 = \sum_{i=1}^{\infty} \mathbf{u}_{1i}, \qquad [10b]$$

$$\mathbf{U}_1 = \sum_{i=1}^{\infty} \mathbf{U}_{1i}, \qquad [10c]$$

where the second subscript under the summation indicates the number of the reflection.

The boundary conditions to be satisfied by the reflected fields are as follows:

The first reflection: at $r = \infty$

$$\mathbf{v}_{11} = 0, \qquad [11a]$$

at $r = a$

$$\mathbf{v}_\infty^* + \mathbf{v}_{11}^* = \mathbf{u}_{11}^*, \qquad [11b]$$

$$\mathbf{u}_{11}^* \cdot \mathbf{t}_r = \mathbf{U}_a \cdot \mathbf{t}_r, \qquad [11c]$$

$$(\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot (\boldsymbol{\pi}^*_{(r)\infty} + \boldsymbol{\pi}^*_{(r)11}) = (\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot \boldsymbol{\tau}^{**}_{(r)11}, \qquad [11d]$$

and in general

$$\mathbf{u}_{1,2k} = \mathbf{U}_{1,2k-1} = 0, \qquad [12a]$$

and

$$\boldsymbol{\tau}_{(r)1,2k} = \mathbf{T}_{(R)1,2K-1} = 0. \qquad [12b]$$

The $2k$th reflection ($k = 1, 2, 3, \ldots$) at $R = \infty$

$$\mathbf{v}_{1,2k} = 0, \qquad [13a]$$

at $R = b$

$$\mathbf{v}^{**}_{1,2k} + \mathbf{v}^{**}_{1,2k-1} = \mathbf{U}^{**}_{1,2k}, \qquad [13b]$$

$$\mathbf{U}^{**}_{1,2k} \cdot \mathbf{t}_R = 0, \qquad [13c]$$

$$(\mathbf{I} - \mathbf{t}_R\mathbf{t}_R) \cdot (\boldsymbol{\pi}^{**}_{(R)1,2k} + \boldsymbol{\pi}^{**}_{(R)1,2k-1}) = (\mathbf{I} - \mathbf{t}_R\mathbf{t}_R)\mathbf{T}^{**}_{(R)1,2k}. \qquad [13d]$$

The $(2k+1)$th reflection at $r = \infty$

$$\mathbf{v}_{1,2k+1} = 0, \qquad [14a]$$

at $r = a$

$$\mathbf{v}^{*}_{1,2k+1} + \mathbf{v}^{*}_{1,2k} = \mathbf{u}^{*}_{1,2k+1}, \qquad [14b]$$

$$\mathbf{u}^{*}_{1,2k+1} \cdot \mathbf{t}_r = 0, \qquad [14c]$$

$$(\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot (\boldsymbol{\pi}^{*}_{(r)1,2k+1} + \boldsymbol{\pi}^{*}_{(r)1,2k}) = (\mathbf{I} - \mathbf{t}_r\mathbf{t}_r) \cdot \boldsymbol{\tau}^{*}_{(r)1,2k+1}. \qquad [14d]$$

Since all the reflected fields satisfy the Stokes' equations of motion, we use Lamb's general solution for Stokesian flow fields as follows:

$$\mathbf{v}_{1,2k} = \sum_{N=1}^{\infty} \left\{ \nabla \times (\mathbf{R}\chi^{1,2k}_{-N-1}) + \nabla\Phi^{1,2k}_{-N-1} - \frac{N-2}{2N(2N-1)} \nabla(R^2 p^{1,2k}_{-N-1}) + \frac{\mathbf{R}}{N} p^{1,2k}_{-N-1} \right\}, \qquad [15a]$$

$$\mathbf{U}_{1,2k} = \sum_{N=1}^{\infty} \left\{ \nabla \times (\mathbf{R}\chi_N{}^{1,2k}) + \nabla\Phi_N{}^{1,2k} + \frac{N+3}{2(N+1)(2N+3)} \nabla(R^2 p_N{}^{1,2k}) - \frac{1}{N+1} \mathbf{R} p_N{}^{1,2k} \right\}, \qquad [15b]$$

$$\mathbf{v}_{1,2k+1} = \sum_{n=1}^{\infty} \left\{ \nabla \times (\mathbf{r}\chi^{1,2k+1}_{-n-1}) + \nabla\Phi^{1,2k+1}_{-n-1} - \frac{n-2}{2n(2n-1)} \nabla(r^2 p^{1,2k+1}_{-n-1}) + \frac{1}{n} \mathbf{r} p^{1,2k+1}_{-n-1} \right\}, \qquad [15c]$$

$$\mathbf{u}_{1,2k+1} = \sum_{n=1}^{\infty} \left\{ \nabla \times (\mathbf{r}\chi_n{}^{1,2k+1}) + \nabla\Phi_n{}^{1,2k+1} + \frac{n+3}{2(n+1)(2n+3)} \nabla(r^2 p_n{}^{1,2k+1}) - \frac{1}{n+1} \mathbf{r} p_n{}^{1,2k+1} \right\}. \qquad [15d]$$

The corresponding pressure fields:

$$p_e{}^{1,2k} = \mu_e \sum_{N=0}^{\infty} p^{1,2k}_{-N-1}, \qquad [16a]$$

$$p_e{}^{1,2k+1} = \mu_e \sum_{n=0}^{\infty} p^{1,2k+1}_{-n-1}, \qquad [16b]$$

$$p_i{}^{1,2k} = \mu_b \sum_{N=1}^{\infty} p_N{}^{1,2k}, \qquad [16c]$$

$$p_i{}^{1,2k+1} = \mu_a \sum_{n=1}^{\infty} p_n{}^{1,2k+1}, \qquad [16d]$$

where $\chi^{1,2k+1}_{-n-1}$, $p^{1,2k+1}_{-n-1}$, $\Phi^{1,2k+1}_{-n-1}$, $\chi^{1,2k}_{-N-1}$, $p^{1,2k}_{-N-1}$, $\Phi^{1,2k}_{-N-1}$, are solid spherical harmonics of degree $(-n-1)$ and $(-N-1)$; $\chi_n{}^{1,2k+1}$, $p_n{}^{1,2k+1}$, $\Phi_n{}^{1,2k+1}$, $\chi_N{}^{1,2k}$, $p_N{}^{1,2k}$, $\Phi_N{}^{1,2k}$, are solid spherical harmonics of degree $n$ and $N$. It is more convenient to evaluate the spherical harmonics by first transforming the boundary conditions (Hetsroni & Haber 1970) as follows:

$$(\mathbf{v}_{1,2k}+\mathbf{v}_{1,2k-1})^{**}\cdot\mathbf{t}_R=\mathbf{U}^{**}_{1,2k}\cdot\mathbf{t}_R=0, \qquad [17a]$$

$$\left(R\frac{\partial v_{R1,2k}}{\partial R}+R\frac{\partial v_{R1,2k-1}}{\partial R}\right)^{**}=\left(R\frac{\partial U_{R1,2k}}{\partial R}\right)^{**}, \qquad [17b]$$

$$[\mathbf{R}\cdot\nabla\times(\mathbf{v}_{1,2k}+\mathbf{v}_{1,2k-1})]^{**}=[\mathbf{R}\cdot\nabla\times\mathbf{U}_{1,2k}]^{**}, \qquad [17c]$$

$$[\mathbf{R}\cdot\nabla\times(\boldsymbol{\pi}_{(R)1,2k}+\boldsymbol{\pi}_{(R)1,2k-1})]^{**}=[\mathbf{R}\cdot\nabla\times\mathbf{T}_{(R)1,2k}]^{**}, \qquad [17d]$$

$$\{\mathbf{R}\cdot\nabla\times[\mathbf{t}_R\times(\boldsymbol{\pi}_{(R)1,2k}+\boldsymbol{\pi}_{(R)1,2k-1})]\}^{**}=\{\mathbf{R}\cdot\nabla\times[\mathbf{t}_R\times\mathbf{T}_{(R)1,2k}]\}^{**}, \qquad [17e]$$

and

$$(\mathbf{v}_{1,2k+1}+\mathbf{v}_{1,2k})^{*}\cdot\mathbf{t}_r=\mathbf{u}^{*}_{1,2k+1}\cdot\mathbf{t}_r=0, \qquad [18a]$$

$$\left(r\frac{\partial v_{r1,2k+1}}{\partial r}+r\frac{\partial v_{r1,2k}}{\partial r}\right)^{*}=\left(r\frac{\partial u_{r1,2k+1}}{\partial r}\right)^{*}, \qquad [18b]$$

$$[\mathbf{r}\cdot\nabla\times(\mathbf{v}_{1,2k+1}+\mathbf{v}_{1,2k})]^{*}=[\mathbf{r}\cdot\nabla\times\mathbf{u}_{1,2k+1}]^{*}, \qquad [18c]$$

$$[\mathbf{r}\cdot\nabla\times(\boldsymbol{\pi}_{(r)1,2k+1}+\boldsymbol{\pi}_{(r)1,2k})]^{*}=[\mathbf{r}\cdot\nabla\times\boldsymbol{\tau}_{(r)1,2k+1}]^{*}, \qquad [18d]$$

$$\{\mathbf{r}\cdot\nabla\times[\mathbf{t}_r\times(\boldsymbol{\pi}_{(r)1,2k+1}+\boldsymbol{\pi}_{(r)1,2k})]\}^{*}=\{\mathbf{r}\cdot\nabla\times[\mathbf{t}_r\times\boldsymbol{\tau}_{(r)1,2k+1}]\}^{*}. \qquad [18e]$$

The solution is now continued by defining the solid spherical harmonies as follows:

$$\chi_N{}^{1,2k}=(-1)^N b^{-N}R^N\sum_{M=0}^{N}C^{1,2k}_{N,M}\cos M\Phi+\hat{C}^{1,2k}_{N,M}\sin M\phi]P_N{}^M(\cos\Theta), \qquad [19a]$$

$$\phi_N{}^{1,2k}=(-1)^N b^{-N+1}R^N\sum_{M=0}^{N}[B^{1,2k}_{N,M}\cos M\Phi+\hat{B}^{1,2k}_{N,M}\sin M\phi]P_N{}^M(\cos\Theta), \qquad [19b]$$

$$p_N{}^{1,2k}=(-1)^N b^{-N-1}R^N\sum_{M=0}^{N}[A^{1,2k}_{N,M}\cos M\Phi+\hat{A}^{1,2k}_{N,M}\sin M\Phi]P_N{}^M(\cos\Theta), \qquad [19c]$$

$$\chi_n{}^{1,2k+1}=-a^{-n}r^n\sum_{m=0}^{n}[c^{1,2k+1}_{n,m}\cos m\phi+\hat{c}^{1,2k+1}_{n,m}\sin m\phi]P_n{}^m(\cos\theta), \qquad [19d]$$

$$\Phi_n{}^{1,2k+1}=-a^{-n+1}r^n\sum_{m=0}^{n}[b^{1,2k+1}_{n,m}\cos m\phi+\hat{b}^{1,2k+1}_{n,m}\sin m\phi]P_n{}^m(\cos\theta), \qquad [19e]$$

$$p_n{}^{1,2k+1}=-a^{-n-1}r^n\sum_{m=0}^{n}[a^{1,2k+1}_{n,m}\cos m\phi+\hat{a}^{1,2k+1}_{n,m}\sin m\phi]P_n{}^m(\cos\theta), \qquad [19f]$$

$$\chi^{1,2k}_{-N-1}=(-1)^N b^{N+1}R^{-N-1}\sum_{M=0}^{N}[C^{1,2k}_{-N-1,M}\cos M\Phi+\hat{C}^{1,2k}_{-N-1,M}\sin M\Phi]P_N{}^M(\cos\Theta), \qquad [19g]$$

$$\Phi^{1,2k}_{-N-1}=(-1)^N b^{N+2}R^{-N-1}\sum_{M=0}^{N}[B^{1,2k}_{-N-1,M}\cos M\Phi+\hat{B}^{1,2k}_{-N-1,M}\sin M\Phi]P_N{}^M(\cos\Theta), \qquad [19h]$$

$$p^{1,2k}_{-N-1}=(-1)^N b^{N}R^{-N-1}\sum_{M=0}^{N}[A^{1,2k}_{-N-1,M}\cos M\Phi+\hat{A}^{1,2k}_{-N-1,M}\sin M\Phi]P_N{}^M(\cos\Theta), \qquad [19i]$$

where $a^{1,2k+1}_{n,m}$, $b^{1,2k+1}_{n,m}$, $c^{1,2k+1}_{n,m}$ are coefficients to be determined.

These coefficients are evaluated by first seeking a recurrance formula, linking the coefficients of a certain reflection to those of a lower order one. This can be readily achieved by noting the similarity between the boundary conditions of [17] and [18] and those of our previous work (with $\mathbf{v}_{1,2k}=\mathbf{V}$, $\mathbf{v}_{1,2k-1}=\mathbf{v}_\infty$, $\mathbf{U}_{1,2k}=\mathbf{v}'$, and $\mathbf{v}_{1,2k+1}=V$, $\mathbf{v}_{1,2k}=\mathbf{v}_\infty$, and $\mathbf{u}_{1,2k+1}=\mathbf{v}'$). It is therefore sufficient to determine the coefficients $\alpha_n{}^m$, $\beta_n{}^m$, and $\gamma_n{}^m$ from the velocity fields $\mathbf{v}_{1,2k}$ and $\mathbf{v}_{1,2k+1}$, by using [21] and [22] of Hetsroni & Haber (1970).

$$\alpha_{n,m}^{1,2k} = \alpha^{n+1} \sum_{N=0}^{\infty} \sum_{M=0}^{N} \beta^N F_{n,m}^{N,M} \frac{n}{2(2n+3)} A_{-N-1,M}^{1,2k}, \qquad [20a]$$

$$\beta_{n,m}^{1,2k} = \alpha^{n-1} \sum_{N=0}^{\infty} \sum_{M=0}^{N} \Big\{ \beta^N F_{n,m}^{N,M} [\beta^2 n B_{-N-1,M}^{1,2k} + A_{-N-1,M}^{1,2k}] \frac{N(2-N)(2n-1) + q(Nn+1-2n-2N)}{2N(2n-1)(2N-1)}$$
$$+ \beta^{N+1} F_{n,m+1}^{N,M} Q C_{-N-1,M}^{1,2k} \Big\}, \qquad [20b]$$

$$\gamma_{n,m}^{1,2k} = \alpha^n \sum_{N=0}^{\infty} \sum_{M=0}^{N} \Big\{ \beta^{N+1} F_{n,m}^{N,M} n N C_{-N-1,M}^{1,2k} + \beta^N \frac{F_{n,m+1}^{N,M}}{N} Q A_{-N-1,M}^{1,2k} \Big\}, \qquad [20c]$$

where

$$F_{n,m}^{N,M} = \begin{cases} (-1)^{1/2(n+m+N-M)} \dfrac{(n+m+N-M)!(n-m+N+M)!}{(n+m)!(N-M)!2^{n+N}[(n-m+N+M)/2]![(n+m+N-M)/2]!} \\ \text{for } (n+m+N-M) \text{ even} \\ 0 \text{ for } (n+m+N-M) \text{ odd} \\ 0 \text{ for } |M-m| > (N+M) \\ 0 \text{ for } M > N, \end{cases} \qquad [21a]$$

$$q = \frac{(n+m)(n+m-1)}{n+m+N-M-1} + \frac{(n-m)(n-m-1)}{(n-m+N+M-1)}, \qquad [21b]$$

$$Q = \frac{(Mn+Nm)(n+m+1)}{n+m+N-M}, \qquad [21c]$$

and

$$\alpha_{N,M}^{1,2k-1} = \beta^{N+1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha^n f_{N,M}^{n,m} \frac{N}{2(2N+3)} a_{-n-1,m}^{1,2k-1}, \qquad [22a]$$

$$\beta_{N,M}^{1,2k-1} = \beta^{N-1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \Big\{ \alpha^n f_{N,M}^{n,m} [\alpha^2 b_{-n-1,m}^{1,2k-1} + a_{-n-1,m}^{1,2k-1}]$$
$$\times \frac{N(2-n)(2N-1) + \Delta(Nn+1-2N-2n)}{2n(2n-1)(2N-1)} + \alpha^{n+1} f_{N,M+1}^{n,m} \delta c_{-n-1,m}^{1,2k-1} \Big\}, \qquad [22b]$$

$$\gamma_{N,M}^{1,2k-1} = \beta^N \sum_{n=0}^{\infty} \sum_{m=0}^{n} \Big\{ \alpha^{n+1} f_{N,M}^{n,m} n N c_{-n-1,m}^{1,2k-1} + \alpha^n \frac{f_{N,M}^{n,m}}{n} \delta a_{-n-1,m}^{1,2k-1} \Big\} \qquad [22c]$$

where

$$f_{N,M}^{n,m} = \begin{cases} (-1)^{1/2(n+m+N-M)} \dfrac{(n+m+N-M)!(n-m+N+M)!}{(n-m_,)!(N+M)!2^{n+N}[(n-m+N+M)/2]![(n+m+N-M)/2]!} \\ \text{for } (n+m+N-M) \text{ even} \\ 0 \text{ for } (n+m+N-M) \text{ odd}, \end{cases} \qquad [23a]$$

$$\Delta = \frac{(N+M)(N+M-1)}{(n-m+N+M-1)} + \frac{(N-M)(N-M-1)}{(n+m+N-M-1)}, \qquad [23b]$$

$$\delta = \frac{(Mn+Nm)(N+M+1)}{(n-m+N+M)}, \qquad [23c]$$

and where

$$\alpha = \frac{a}{l}; \qquad \beta = \frac{b}{l}. \qquad [24]$$

The solution is continued following the scheme we used previously, viz.

$$A^{1,2k}_{-N-1,M} = \frac{2N-1}{(N+1)(1+\lambda_b)}\{\lambda_b[(2N+3)\alpha^{1,2k-1}_{N,M} + (2N+1)\beta^{1,2k-1}_{N,M}] + 2\beta^{1,2k-1}_{N,M}\}, \qquad [25a]$$

$$B^{1,2k}_{-N-1,M} = \frac{1}{2(N+1)(1+\lambda_b)}\{\lambda_b[(2N+1)\alpha^{1,2k-1}_{N,M} + (2N-1)\beta^{1,2k-1}_{N,M}] - 2\alpha^{1,2k-1}_{N,M}\}, \qquad [25b]$$

$$A^{1,2k}_{N,M} = -\frac{(2N-1)(2N+3)}{N(1+\lambda_b)}\left[\frac{2N+3}{2N+1}\alpha^{1,2k-1}_{N,M} + \beta^{1,2k-1}_{N,M}\right], \qquad [25c]$$

$$B^{1,2k}_{N,M} = -\frac{A^{1,2k}_{N,M}}{2(2N+3)}, \qquad [25d]$$

$$C^{1,2k}_{-N-1,M} = \frac{(N-1)(1-\lambda_b)}{N(N+1)[(N-1)\lambda_b + N + 2]}\gamma^{1,2k-1}_{N,M}, \qquad [25e]$$

$$C^{1,2k}_{N,M} = \frac{2N+1}{N(N+1)[(N-1)\lambda_b + N + 2]}\gamma^{1,2k-1}_{N,M}, \qquad [25f]$$

$$a^{1,2k+1}_{-n-1,m} = \frac{2n-1}{(n+1)(1+\lambda_a)}\{\lambda_a[(2n+3)\alpha^{1,2k}_{n,m} + (2n+1)\beta^{1,2k}_{n,m}] + 2\beta^{1,2k}_{n,m}\}, \qquad [26a]$$

$$b^{1,2k+1}_{-n-1,m} = \frac{1}{2(n+1)(1+\lambda_a)}\{\lambda_a[(2n+1)\alpha^{1,2k}_{n,m} + (2n-1)\beta^{1,2k}_{n,m}] - 2\alpha^{1,2k}_{n,m}\}, \qquad [26b]$$

$$a^{1,2k+1}_{n,m} = -\frac{(2n-1)(2n+3)}{n(1+\lambda_a)}\left[\frac{(2n+3)}{2n+1}\alpha^{1,2k}_{n,m} + \beta^{1,2k}_{n,m}\right], \qquad [26c]$$

$$b^{1,2k+1}_{n,m} = -\frac{a^{1,2k}_{n,m}}{2(2n+3)}, \qquad [26d]$$

$$c^{1,2k+1}_{-n-1,m} = \frac{(n-1)(1-\lambda_a)}{n(n+1)[(n-1)\lambda_a + n + 2]}\gamma^{1,2k}_{n,m}, \qquad [26e]$$

$$c^{1,2k+1}_{n,m} = \frac{2n+1}{n(n+1)[(n-1)\lambda_a + n + 2]}\gamma^{1,2k}_{n,m}, \qquad [26f]$$

for $k = 1, 2, 3 \ldots$.

Thus, the solution of the reflection $2k$ was expressed in terms of the coefficients of reflection $(2k-1)$, and the $(2k+1)$th reflection was expressed in terms of the coefficients of the reflection $2k$.

Obviously, it will suffice to solve the coefficients of the first reflection only, namely $\mathbf{v}_{11}$ and $\mathbf{u}_{11}$. This is achieved by defining

$$\mathbf{v}_{1,0} = \mathbf{v}_\infty - \mathbf{u}_a,$$

and then applying [26] with $k = 0$. This is quite permissible since the boundary conditions of [11] are similar to [14] for $k = 0$, with the exception that $\mathbf{u}_a$ subtracted from $\mathbf{u}_{11}$.

The coefficients $\alpha^{1,0}_{n,m}$, $\beta^{1,0}_{n,m}$, and $\gamma^{1,0}_{n,m}$ are determined in an analogous way to the one described by Hetsroni & Haber (1970), their [17], [19], [20], with $\alpha^{1,0}_{n,m} = \alpha_m{}^n$, etc.

A similar solution can be obtained for the flow fields $\mathbf{v}_2$, $\mathbf{u}_2$, and $\mathbf{U}_2$.

Thus, a solution for the flow fields is obtained to any desired accuracy which is determined by the order of $\alpha$ and $\beta$, for a given $\mathbf{v}_\infty$. Also, the drag forces acting on the droplets is calculated:

$$\mathbf{f}_{Da} = 4\pi\mu_e a \sum_{k=0}^{\infty} [(a_{-2,1}^{1,2k+1} + a_{-2,1}^{2,2k+1})\mathbf{i} + (\hat{a}_{-2,1}^{1,2k+1} + \hat{a}_{-2,1}^{2,2k+1})\mathbf{j} + (a_{-2,0}^{1,2k+1} + a_{-2,0}^{2,2k+1})\mathbf{k}], \quad [27a]$$

$$\mathbf{F}_{Db} = 4\pi\mu_e b \sum_{k=0}^{\infty} [(A_{-2,1}^{1,2k} + A_{-2,1}^{2,2k})\mathbf{i} + (\hat{A}_{-2,1}^{1,2k} + \hat{A}_{-2,1}^{2,2k})\mathbf{j} + (A_{-2,0}^{1,2k} + A_{-2,0}^{2,2k})\mathbf{k}]. \quad [27b]$$

*Examples*

As a first example we present the solution for the drag forces and terminal settling velocities of two droplets falling in an unbounded quiescent fluid in a gravitational field. The solution is carried out to $0(\alpha^n\beta^m)$ where $n + m \leq 5$.

The velocities of the droplets are expressed by:

$$\mathbf{u}_a = u_{ax}\mathbf{i} + u_{az}\mathbf{k}, \quad [28a]$$

$$\mathbf{U}_b = U_{bx}\mathbf{i} + U_{bz}\mathbf{k}, \quad [28b]$$

and

$$\mathbf{v}_\infty = 0. \quad [28c]$$

From the following vectorial operations we obtain:

$$\begin{aligned}\mathbf{t}_r \cdot (\boldsymbol{v}_\infty - \boldsymbol{u}_a) = \mathbf{t}_r \cdot \mathbf{v}_{1,0} &= -[u_{ax} \sin\theta \cos\phi + u_{az} \cos\theta \\ &= -(u_{ax} \cos\phi P_1{}^1(\cos\theta) + u_{az} P_1(\cos\theta))],\end{aligned} \quad [29a]$$

$$\mathbf{t}_r \cdot \nabla \times (\mathbf{v}_\infty - \mathbf{u}_a) = 0, \quad [29b]$$

using [21] and [22] from Hetsroni & Haber (1970):

$$\beta_{1,0}^{1,0} = -u_{az}, \quad [29c]$$

$$\beta_{1,1}^{1,0} = -u_{ax}, \quad [29d]$$

and all the other coefficients are zero. Substituting [29] into [26] one obtains the coefficient $a_{-2,0}^{1,1}$, $b_{-2,0}^{1,1}$, $a_{1,0}^{1,1}$, $b_{1,0}^{1,1}$, and $a_{-2,1}^{1,1}$, $b_{-2,1}^{1,1}$, $a_{1,1}^{1,1}$, and $b_{1,1}^{1,1}$ where only $a_{-2,0}^{1,1}$ and $a_{-2,1}^{1,1}$ are significant for the calculation of the drag forces. However, in order to bring the solution to the desired accuracy and order of $(n + m)$, one must compute higher reflections. Using [25] and [26] up to the sixth reflection, the desired order is obtained. All higher reflections have no contribution to that order.

An identical procedure should be followed for solving $\mathbf{v}_2$. However, since $\mathbf{v}_2$ is completely similar to $\mathbf{v}_1$, solution for the latter is applicable to the $\mathbf{v}_2$ field, with '$b$' replacing '$a$', $\lambda_b$ replacing $\lambda_a$, $U_b$ replacing $u_a$ and $\beta$ replacing $\alpha$, and *vice versa*.

After some lengthy algebra the following is obtained:

$$\begin{aligned}\frac{F_{az}}{6\pi\mu_e a \dfrac{2/3 + \lambda_a}{1 + \lambda_a}} &= u_{az}\left[1 + \frac{(2 + 3\lambda_a)(2 + 3\lambda_b)}{4(1 + \lambda_a)(1 + \lambda_b)}\alpha\beta + \frac{(2 + 3\lambda_a)(2 + 3\lambda_b)}{4(1 + \lambda_a)(1 + \lambda_b)}\beta^3\alpha\right. \\ &\quad \left. - \frac{\lambda_a(2 + 3\lambda_b)}{2(1 + \lambda_a)(1 + \lambda_b)}\alpha^3\beta + \frac{(2 + 3\lambda_a)^2(2 + 3\lambda_b)^2}{16(1 + \lambda_a)^2(1 + \lambda_b)}\alpha^2\beta^2\right] \\ &\quad + U_{bz}\left[-\frac{2 + 3\lambda_b}{2(1 + \lambda_b)}\beta + \frac{\lambda_b}{2(1 + \lambda_b)}\beta^3\right. \\ &\quad \left. + \frac{\lambda_a(2 + 3\lambda_b)}{2(2 + 3\lambda_a)(1 + \lambda_b)}\beta\alpha^2 - \frac{(2 + 3\lambda_a)(2 + 3\lambda_b)^2}{8(1 + \lambda_a)(1 + \lambda_b)^2}\beta^2\alpha\right] \\ &\equiv K_{az}(\alpha, \beta, \lambda_a, \lambda_b)u_{az} + K_{bz}(\alpha, \beta, \lambda_a, \lambda_b)U_{bz},\end{aligned} \quad [30]$$

$$\frac{F_{ax}}{6\pi\mu_e a\dfrac{2/3+\lambda_a}{1+\lambda_a}} = u_{ax}\left[1+\frac{(2+3\lambda_a)(2+3\lambda_b)}{16(1+\lambda_a(1+\lambda_b)}\alpha\beta+\frac{(2+3\lambda_a)(2+3\lambda_b)}{8(1+\lambda_a)(1+\lambda_b)}\beta^3\alpha\right.$$

$$\left.+\frac{\lambda_a(2+3\lambda_b)}{8(1+\lambda_a)(1+\lambda_b)}\alpha^3\beta+\frac{(2+3\lambda_a)^2(2+3\lambda_b)^2}{256(1+\lambda_a)^2(1+\lambda_b)^2}\alpha^2\beta^2\right]$$

$$-U_{bx}\left[\frac{2+3\lambda_b}{4(1+\lambda_b)}\beta+\frac{\lambda_b}{4(1+\lambda_b)}\beta^3+\frac{\lambda_a(2+3\lambda_b)}{4(2+3\lambda_a)(1+\lambda_b)}\beta\alpha^2\right.$$

$$+\frac{(2+3\lambda_a)(2+3\lambda_b)^2}{64(1+\lambda_a)(1+\lambda_b)^2}\beta^2\alpha+\frac{7(2+3\lambda_a)(2+3\lambda_b)}{64(1+\lambda_a)(1+\lambda_b)}\alpha\beta^4$$

$$\left.+\frac{7(2+3\lambda_b)^2}{64(1+\lambda_b)^2}\alpha^3\beta^2+\frac{(2+3\lambda_a)^2(2+3\lambda_b)^3}{1024(1+\lambda_a)^2(1+\lambda_b)^3}\alpha^2\beta^3\right]$$

$$\equiv K_{ax}(\alpha,\beta,\lambda_a,\lambda_b)u_{ax}+K_{bx}(\alpha,\beta,\lambda_a,\lambda_b)U_{bx}. \qquad [31]$$

Similar solutions are obtained for the drag force acting on droplet $b$. Thus,

$$\frac{F_{bx}}{6\pi\mu_e b\left(\dfrac{2/3+\lambda_b}{1+\lambda_b}\right)} = L_{bx}(\alpha,\beta,\lambda_a,\lambda_b)U_{bx}+L_{ax}(\alpha,\beta,\lambda_a,\lambda_b)u_{ax}, \qquad [32]$$

and

$$\frac{F_{bz}}{6\pi\mu_e b\left(\dfrac{2/3+\lambda_b}{1+\lambda_b}\right)} = L_{bz}(\alpha,\beta,\lambda_a,\lambda_b)U_{bz}+L_{az}(\alpha,\beta,\lambda_a,\lambda_b)u_{az}. \qquad [33]$$

It is obvious that

$$\left.\begin{aligned} L_{bx}(\alpha,\beta,\lambda_a,\lambda_b) &= K_{ax}(\beta,\alpha,\lambda_b,\lambda_a),\\ L_{ax}(\alpha,\beta,\lambda_a,\lambda_b) &= K_{bx}(\beta,\alpha,\lambda_b,\lambda_a),\\ L_{bz}(\alpha,\beta,\lambda_a,\lambda_b) &= K_{az}(\beta,\alpha,\lambda_b,\lambda_a),\\ \text{and}\quad L_{az}(\alpha,\beta,\lambda_a,\lambda_b) &= K_{bz}(\beta,\alpha,\lambda_b,\lambda_a). \end{aligned}\right\} \qquad [34]$$

A very simple relationship exists between $K_{bx}$, $L_{ax}$ and $K_{bz}$, $L_{az}$, namely

$$\left.\begin{aligned} \frac{2/3+\lambda_a}{1+\lambda_a}\alpha K_{bx}(\alpha,\beta,\lambda_a,\lambda_b) &= \frac{2/3+\lambda_b}{1+\lambda_b}\beta L_{ax}(\alpha,\beta,\lambda_a,\lambda_b),\\ \frac{2/3+\lambda_a}{1+\lambda_a}\alpha K_{bz}(\alpha,\beta,\lambda_a,\lambda_b) &= \frac{2/3+\lambda_b}{1+\lambda_b}\beta L_{az}(\alpha,\beta,\lambda_a,\lambda_b). \end{aligned}\right\} \qquad [35]$$

Hence

$$\left.\begin{aligned} \frac{2/3+\lambda_a}{1+\lambda_a}\alpha K_{bx}(\alpha,\beta,\lambda_a,\lambda_b) &= \frac{2/3+\lambda_b}{1+\lambda_b}\beta K_{bx}(\beta,\alpha,\lambda_b,\lambda_a),\\ \frac{2/3+\lambda_a}{1+\lambda_a}\alpha K_{bz}(\alpha,\beta,\lambda_a,\lambda_b) &= \frac{2/3+\lambda_b}{1+\lambda_b}\beta K_{bz}(\beta,\alpha,\lambda_b,\lambda_a). \end{aligned}\right\} \qquad [36]$$

Equations [35] can be proved by substitution into [30] and [31]. A general proof to any order of accuracy of the $K$'s is presented in appendix A.

These solutions contain as special case the solution for rigid spheres moving in an unbounded gravitational velocity field presented in Happel & Brenner (1965).

In order to obtain the velocities of the droplets, simple force balance is used since:

$$\left.\begin{aligned}\mathbf{F}_{ga} &= \frac{4}{3}\pi a^3\Delta\rho_a\mathbf{g} = 6\pi\mu_e a\mathbf{U}_{0a}\frac{2/3+\lambda_a}{1+\lambda_a},\\ &\text{and}\\ \mathbf{F}_{gb} &= \frac{4}{3}\pi b^3\Delta\rho_b\mathbf{g} = 6\pi\mu_e b\mathbf{U}_{0b}\frac{2/3+\lambda_b}{1+\lambda_b},\end{aligned}\right\} \quad [37]$$

where $\mathbf{U}_{0a}$ and $\mathbf{U}_{0b}$ are the terminal settling velocities of drop $a$ and $b$, respectively, suspended in an unbounded quiescent gravitational field, then:

$$\left.\begin{aligned}K_{az}u_{az} + K_{bz}U_{bz} &= \mathbf{U}_{0a}\cdot\mathbf{k},\\ L_{az}u_{az} + L_{bz}U_{bz} &= \mathbf{U}_{0b}\cdot\mathbf{k},\\ K_{ax}u_{ax} + K_{bx}U_{bx} &= \mathbf{U}_{0a}\cdot\mathbf{i},\\ L_{ax}u_{ax} + L_{bx}U_{bx} &= \mathbf{U}_{0b}\cdot\mathbf{i}.\end{aligned}\right\} \quad [38]$$

The solution of [38] yields:

$$\mathbf{u}_a = u_{ax}\mathbf{i} + u_{az}\mathbf{k} = [k_{az}\mathbf{kk} + k_{ax}(\mathbf{I}-\mathbf{kk})]\cdot\mathbf{U}_{01} + [k_{bz}\mathbf{kk} + k_{bx}(\mathbf{I}-\mathbf{kk})]\cdot\mathbf{U}_{02},$$

$$\mathbf{U}_b = U_{bx}\mathbf{i} + U_{bz}\mathbf{k} = [l_{az}\mathbf{kk} + l_{ax}(\mathbf{I}-\mathbf{kk})]\cdot\mathbf{U}_{01} + [l_{bz}\mathbf{kk} + l_{bx}(\mathbf{I}-\mathbf{kk})]\cdot\mathbf{U}_{02},$$

where

$$k_{az} = \frac{L_{bz}}{\Delta_z}; \qquad k_{bz} = -\frac{K_{bz}}{\Delta_z},$$

$$l_{az} = -\frac{L_{az}}{\Delta_z}; \qquad l_{bz} = \frac{K_{az}}{\Delta_z},$$

$$\Delta_z = K_{az}L_{bz} - K_{bz}L_{az},$$

and $k_{ax}$, $k_{bx}$, $l_{ax}$, $l_{bx}$ are obtained similarly by interchanging the subscript $z$ by $x$.

*Example* 2

In the second example we present the solution for the drag force acting on two drops submerged in a Couette flow field.

The undisturbed velocity field is $\mathbf{v}_\infty = GX\mathbf{k}$ (figure 1).

Since $v_\infty^* = G\mathbf{k}[X]^* = G\mathbf{k}(a\sin\theta\cos\phi - l_a)$ and $v_\infty^{**} = G\mathbf{k}[X]^{**} = G\mathbf{k}(b\sin\Theta\cos\Phi - l_b)$,

$$\Pi^*_{(r)\infty} = -p^*\mathbf{t}_r + \mu G(\mathbf{k}\sin\theta\cos\phi + \mathbf{i}\cos\theta),$$

$$\Pi^{**}_{(R)\infty} = -p_\infty^{**}\mathbf{t}_R + \mu G(\mathbf{k}\sin\Theta\cos\Phi + \mathbf{i}\cos\Theta).$$

The boundary conditions for $\mathbf{v}_1$ and $\mathbf{v}_2$ are similar. We therefore limit ourselves to the solution of $\mathbf{v}_1$, $\mathbf{u}_1$ and $\mathbf{U}_1$ and the solution for $\mathbf{v}_2$, $\mathbf{u}_2$ and $\mathbf{U}_2$ is easily obtained by interchanging $l_a$ with $l_b$, $a$ with $b$ and $\lambda_a$ with $\lambda_b$ and *vice versa*. Further simplification can be achieved by dividing $\mathbf{v}_\infty$ into two terms:

$$\mathbf{v}_\infty = -Gl_a\mathbf{k} + Gr\sin\theta\cos\phi\mathbf{k} = \mathbf{v}'_\infty + \mathbf{v}''_\infty.$$

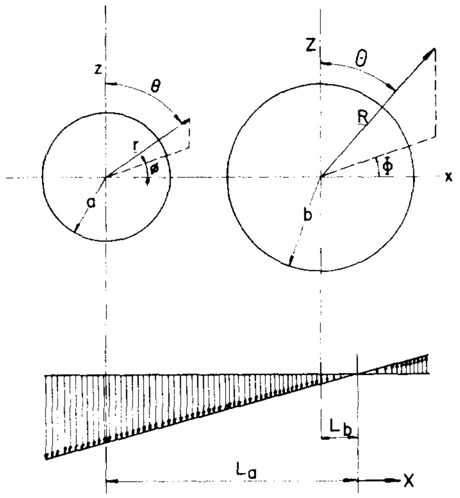


Figure 1.

A solution for the first term $\mathbf{v}'_\infty$ has already been obtained in example 1 where one has to interchange in [31] $u_a$ with $Gl_a$ and $U_b$ with $Gl_b$. The second term is a new one and the solution for the drag force is presented as follows:

From the following vectorial operations, we obtain:

$$\mathbf{t}_r \cdot \mathbf{v}''_\infty = Gr \sin\theta \cos\theta \cos\phi = 1/3Ga\left(\frac{r}{a}\right)P_2{}^1(\cos\theta)\cos\phi,$$

$$\mathbf{r} \cdot \nabla x \mathbf{v}''_\infty = -Gr\sin\theta\sin\phi = -Ga\left(\frac{r}{a}\right)P_1{}^1(\cos\theta)\sin\phi.$$

Using [21] and [22] from Hetsroni & Haber (1970): $\beta^{1,0}_{2,1} = 1/3Ga$, $\hat{\gamma}^{1,0}_{1,1} = -Ga$. To obtain a solution of $0(\alpha^n\beta^m)$ where $n + m \leq 7$ the following coefficients were calculated (five reflections are needed):

1st reflection—$a^{1,1}_{-3,1}$, $b^{1,1}_{-3,1}$, $\alpha^{1,1}_{1,0}$, $\beta^{1,1}_{1,0}$, $\gamma^{1,1}_{1,0}$, $\alpha^{1,1}_{2,1}$, $\beta^{1,1}_{2,1}$, $\gamma^{1,1}_{2,1}$, $\alpha^{1,1}_{3,0}$, $\beta^{1,1}_{3,0}$, $\alpha^{1,1}_{3,2}$, $\beta^{1,1}_{3,2}$,

2nd reflection—$A^{1,2}_{-2,0}$, $A^{1,2}_{-3,1}$, $B^{1,2}_{-2,0}$, $B^{1,2}_{-3,1}$, $C^{1,2}_{-3,1}$, $A^{1,2}_{-4,0}$, $A^{1,2}_{-4,2}$, $B^{1,2}_{-4,0}$, $B^{1,2}_{-4,2}$, $\alpha^{1,2}_{1,0}$, $\beta^{1,2}_{1,0}$, $\gamma^{1,2}_{1,0}$, $\alpha^{1,2}_{1,1}$, $\beta^{1,2}_{1,1}$,

3rd reflection—$a^{1,3}_{-2,0}$, $a^{1,3}_{-2,1}$, $\beta^{1,3}_{1,0}$,

4th reflection—$A^{1,4}_{-2,0}$, $\beta^{1,4}_{1,0}$,

5th reflection—$a^{1,5}_{-2,0}$.

The drag force acting on the drops is calculated by [27]

$$f^1_{\mathrm{D}a} = 4a\pi\mu_e\{(a^{1,3}_{-2,0} + a^{1,5}_{-2,0})\mathbf{k} + a^{1,3}_{-2,1}\mathbf{i}\} + 0(\alpha^n\beta^m)(n+m) > 7,$$

$$F^1_{\mathrm{D}b} = 4\pi b\mu_e(A^{1,2}_{-2,0} + A^{1,4}_{-2,0})\mathbf{k} + 0(\alpha^n\beta^m)(m+n) > 7.$$

By interchanging $a$, $\alpha$, $\lambda_a$ with $b$, $\beta$, $\lambda_b$ and *vice versa*, $\mathbf{f}^2_{\mathrm{D}a}$ and $F^2_{\mathrm{D}b}$ are obtained. Hence the total drag force acting on a drop $a$ is as follows:

$$f_{\mathrm{D}a} = \left\{-4\pi\mu_e Ga^2\left[\alpha^4\beta\frac{\lambda_a(2+3\lambda_a)(2+3\lambda_b)}{16(1+\lambda_a)^2(1+\lambda_b)} + \alpha^2\beta^3\frac{\lambda_b(2+3\lambda_a)(2+5\lambda_a)}{16(1+\lambda_a)^2(1+\lambda_b)}\right.\right.$$

$$+\alpha^6\beta\frac{\lambda_a^2(2+3\lambda_b)}{16(1+\lambda_a)^2(1+\lambda_b)}+\alpha^2\beta^5\frac{(2+3\lambda_a)(2+5\lambda_a)(6+37\lambda_a)}{384(1+\lambda_a)^2(1+7\lambda_b)}$$

$$-\beta^3\alpha^4\frac{\lambda_a(17\lambda_a\lambda_b)+10\lambda_a+6\lambda_b+4)}{16(1+\lambda_a)^2(1+\lambda_b)}+\alpha^5\beta^2\frac{\lambda_a(2+3\lambda_a)^2(2+3\lambda_b)^2}{256(1+\lambda_a)^3(1+\lambda_b)^2}$$

$$+\alpha^3\beta^4\frac{\lambda_b(2+3\lambda_b)(2+3\lambda_a)(2+5\lambda_a)}{256(1+\lambda_a)^2(1+\lambda_b)^2}\Bigg]+4\pi\mu_e Gab\Bigg[\frac{\lambda_b(2+3\lambda_a)}{4(1+\lambda_a)(1+\lambda_b)}\beta^4$$

$$+\frac{\lambda_a(2+5\lambda_b)}{4(1+\lambda_a)(1+\lambda_b)}\alpha^2\beta^2+\beta^5\alpha\frac{\lambda_b(2+3\lambda_b)(2+3\lambda_a)^2}{64(1+\lambda_a)^2(1+\lambda_b)^2}+\alpha^3\beta^3\frac{\lambda_a(2+3\lambda_a)(2+3\lambda_b)(2+5\lambda_b)}{64(1+\lambda_a)^2(1+\lambda_b)^2}\Bigg]\Bigg\}\mathbf{k}$$

$$+\left\{\frac{5\pi\mu_e}{6}Ga^2\frac{(1+\lambda_b)(2+5\lambda_a)}{(4+\lambda_b)(1+\lambda_a)}\beta^5\alpha^2\right\}\mathbf{i}+0(\alpha^n\beta^m)(m+n)>7,$$

and a similar expression is obtained for $\mathbf{F}_{Db}$.

The result is interesting since a drag force was obtained in the $\mathbf{i}$ direction. This force causes the drops to migrate along their line of centers perpendicular to the direction of $\mathbf{v}_\infty$. The direction of the motion depends on the sign of $G$ and the magnitude of $\lambda_a$ and $\lambda_b$ (smaller or greater than unity).

No such component of the drag force is obtained from the solution of $\mathbf{v}'_\infty$. The magnitude of that velocity depends mainly on the radii of the drops, the distance between their centers and the shear intensity. It is well understood that when the drops come closer the first term above cannot explain the whole phenomenon. More reflections, or even an exact solution are needed.

## 4. SUMMARY

Two solutions are presented in the paper. One for the problem of two drops of different size and viscosity moving in an arbitrary unbounded field along their line of centers, and the other for two drops moving perpendicular to their line of centers. The reflection method is used and recursive formulae obtained which lead to solutions for the flow fields to any desired accuracy.

As special cases, approximations for the drag force, to the order of $(a/l)^m(b/l)^n$ (when $m+n<5$), and to the settling velocities are obtained for two drops moving in a quiescent gravitational unbounded field, and in Couette flow.

The coefficients $K_{az}$ and $K_{bz}$, defined in [30], are the coefficients for the drag force acting on drop a induced by the motion of drops $a$ and $b$, respectively, in the $z$ direction (problem I).

The coefficients $K_{ax}$ and $K_{bx}$, defined in [31], are the coefficients for the drag force acting on the drop $a$ induced by the motion of drops $a$ and $b$, respectively, in the $x$ direction (problem II).

The drag force coefficients $K_{az}$, $K_{bz}$ are plotted in figures 2 to 5 and compared with the exact

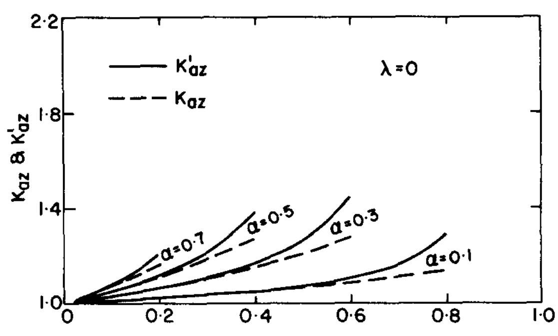


Figure 2. $K_{az}$ and $K'_{az}$ are the respective approximate [30] and exact (Haber *et al.* 1974) coefficients of the drag force acting on drop $a$ induced by the motion of drop $a$ in the $z$ direction. ($\lambda = 0$, air bubbles in water).

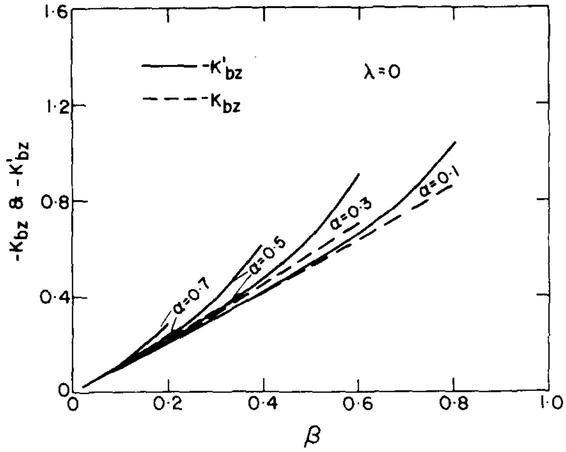


Figure 3. $K_{bz}$ and $K'_{bz}$ are the respective approximate [30] and exact (Haber *et al.* 1974) coefficients of the drag force acting on drop *a* induced by the motion of drop *b* in the *z* direction. ($\lambda = 0$, air bubbles in water).

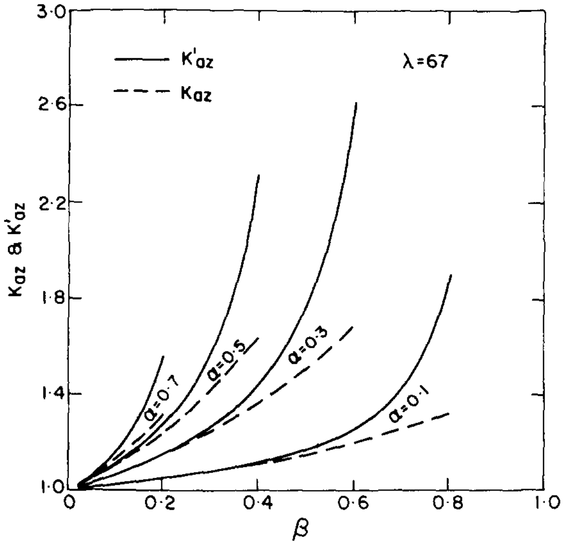


Figure 4. $K_{az}$ and $K'_{az}$ are the respective approximate [30] and exact (Haber *et al.* 1974) coefficients of the drag force acting on drop *a* induced by the motion of drop *a* in the *z* direction. ($\lambda = 67$, water drops in air).

drag coefficients $K'_{az}$ and $K'_{bz}$ obtained by Haber *et al.* (1974). Two cases are plotted: $\lambda = 0$ (approximating air bubbles in water) and $\lambda = 67$ (approximating water drops in air). The results obtained by the method of reflection are good for the $\lambda = 0$ and become worse as $\lambda$ increases. For $\alpha + \beta$ close to unity (two drops close to each other) the method of reflection fails entirely. Instead of increasing to infinity the coefficients $K_{az}$ and $K_{bz}$ are limited constants of order of magnitude 1. The method of reflection yields accurate results for $\alpha + \beta < 0.5$.

Exact drag force coefficients for two drops moving perpendicular to their line of centers are yet unknown and the results obtained for $K_{ax}$ and $K_{bx}$ (see figures 6 and 7) can be compared only with the ones obtained by Happel & Brenner (1965) for the *solid* spheres. The results plotted for $\lambda = 67$ were compared with the drag coefficients obtained for $\lambda = \infty$ and the results obtained by Happel & Brenner (1965). No differences of any significance were observed.

Other interesting results are that the coefficients $K_{az}$ and $K_{bz}$ are larger than $K_{ax}$ and $K_{bx}$, respectively, (for the same $\alpha$, $\beta$, and $\lambda$) as well as the expected result that all the coefficients grow monotonically as $\lambda$ increases.

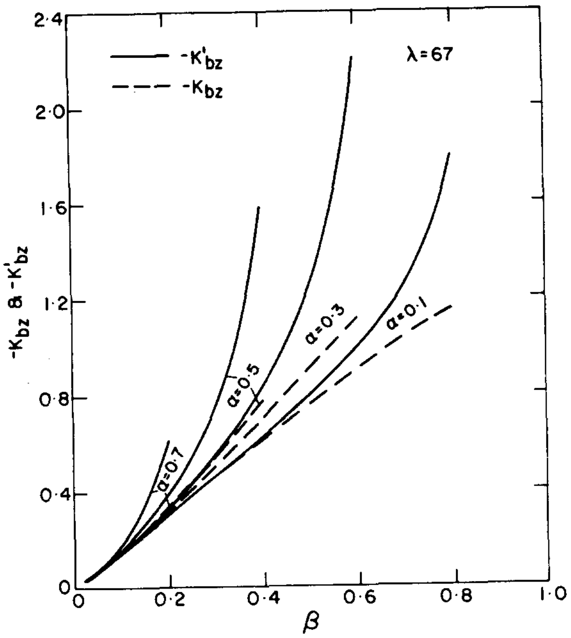


Figure 5. $K_{bz}$ and $K'_{bz}$ are the respective approximate [30] and exact (Haber *et al.* 1974) coefficients of the drag force acting on drop *a* induced by the motion of drop *b* in the *z* direction ($\lambda = 67$, water drops in air).

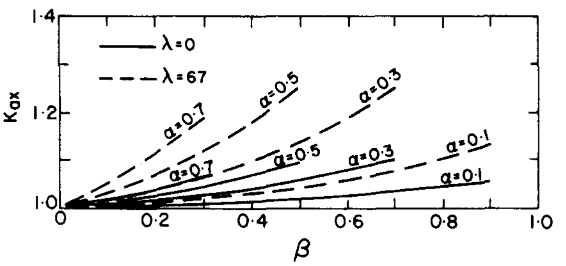


Figure 6. $K_{ax}$ is the approximate coefficient [31] of the drag force acting on drop *a* induced by the motion of drop *a* in the *x* direction.

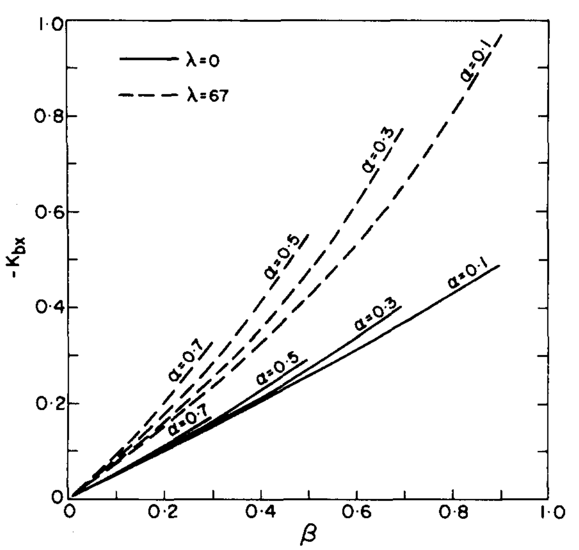


Figure 7. $K_{bx}$ is the approximate coefficient [31] of the drag force acting on drop *a* induced by the motion of drop *b* in the *x* direction.

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## APPENDIX A

Assume a system of two rigid spheres moving in an unbounded quiescent fluid.

We shall describe two different problems.

(I) A sphere of radius $a$ is moving at velocity $\mathbf{u}_a$ while the sphere of radius $b$ is at rest. The velocity field and the stress field generated are $\mathbf{u}_1$ and $\boldsymbol{\pi}_1$, respectively.

(II) A sphere of radius $b$ is moving at velocity $\mathbf{U}_b$ while the sphere of radius $a$ is at rest. The velocity field and the stress fields generated are $\mathbf{u}_2$ and $\boldsymbol{\pi}_2$, respectively.

Using the reciprocal theorem one obtains:

$$\int_S \mathbf{u}_1 \cdot \boldsymbol{\pi}_2 \cdot \mathbf{ds} = \int_S \mathbf{u}_2 \cdot \boldsymbol{\pi}_2 \cdot \mathbf{ds}, \qquad [\mathrm{A1}]$$

where $S$ is a surface which contains any arbitrary closed volume. Assume that $S$ consists of three different surfaces: $S_\infty$, which is a surface of a sphere with a very large radius $r$ and which contains the two spheres; $S_a$ and $S_b$, which are the surfaces of spheres $a$ and $b$, respectively. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ decrease asymptotically with $r^{-1}$, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ with $r^{-2}$, $\overline{ds}$ with $r^2$ and $\mathbf{u}_1 \cdot \boldsymbol{\pi}_2 \cdot \mathbf{ds}$ and $\mathbf{u}_2 \cdot \boldsymbol{\pi}_1 \cdot \mathbf{ds}$ with $r^{-1}$, the integrals [A1] on the surface $S_\infty$ are negligible. Since $\mathbf{u}_1 = \mathbf{u}_a$ on $S_a$, and $\mathbf{u}_1 = 0$ on $S_b$, and since $\mathbf{u}_2 = \mathbf{U}_b$ on $S_b$ and $\mathbf{u}_2 = 0$ on $S_a$, one obtains:

$$\mathbf{u}_a \cdot \int_{S_a} \boldsymbol{\pi}_2 \cdot \mathbf{ds} = \mathbf{U}_b \cdot \int_{S_b} \boldsymbol{\pi}_1 \cdot \mathbf{ds}. \qquad [\mathrm{A2}]$$

But

$$\int_{S_a} \boldsymbol{\pi}_2 \cdot \mathbf{ds} = \mathbf{F}_a^{(b)},$$

$$\int_{S_b} \boldsymbol{\pi}_1 \cdot \mathbf{ds} = \mathbf{F}_b^{(a)},$$

where $\mathbf{F}_a^{(b)}$ is the drag force acting on sphere $a$ induced by the motion of sphere $b$ moving at the velocity $\mathbf{U}_b$ and similarly for $\mathbf{F}_b^{(a)}$. Hence

$$\mathbf{u}_a \cdot \mathbf{F}_a^{(b)} = \mathbf{U}_b \cdot \mathbf{F}_b^{(a)}. \qquad [A3]$$

$\mathbf{F}_a^{(b)}$ and $\mathbf{F}_b^{(a)}$ can be written in the following general form:

$$\left.\begin{aligned} \mathbf{F}_a^{(b)} &= 6\pi\mu_e a \boldsymbol{K}_b \cdot \mathbf{U}_b, \\ \mathbf{F}_b^{(a)} &= 6\pi\mu_e b \boldsymbol{K}_a \cdot \mathbf{u}_a, \end{aligned}\right\} \qquad [A4]$$

where $\mathbf{K}_a$ and $\mathbf{K}_b$ are second rank tensors.

Substituting [A4] in [A3]:

$$a\mathbf{u}_a \cdot \boldsymbol{K}_b \cdot \mathbf{U}_b = b\mathbf{U}_b \cdot \boldsymbol{K}_a \cdot \mathbf{u}_a.$$

But

$$\mathbf{U}_b \cdot \mathbf{K}_a \cdot \boldsymbol{u}_a = \mathbf{U}_b \cdot (\mathbf{u}_a \cdot \boldsymbol{K}_a^T) = \mathbf{u}_a \cdot \boldsymbol{K}_a^T \cdot \mathbf{U}_b.$$

Hence,

$$a\mathbf{u}_a \cdot \boldsymbol{K}_b \cdot \mathbf{U}_b = b\mathbf{u}_a \cdot \boldsymbol{K}_a^T \cdot \mathbf{U}_b,$$

and since $\mathbf{u}_a$ and $\mathbf{U}_b$ were arbitrarily chosen

$$a\boldsymbol{K}_b = b\boldsymbol{K}_a^T. \qquad [A5]$$

But $\boldsymbol{K}_a$ and $\boldsymbol{K}_b$ are expressed in the $(x, y, z)$ coordinate system as follows:

$$\boldsymbol{K}_a = \left\{\begin{matrix} L_{ax} & 0 & 0 \\ 0 & L_{ay} & 0 \\ 0 & 0 & L_{az} \end{matrix}\right\}, \qquad \boldsymbol{K}_b = \left\{\begin{matrix} K_{bx} & 0 & 0 \\ 0 & K_{by} & 0 \\ 0 & 0 & K_{bz} \end{matrix}\right\}.$$

But, since $K_{bx} = K_{by}$ and $L_{ax} = L_{ay}$, then

$$\begin{aligned} aK_{bx} &= bL_{ax}, \\ aK_{bz} &= bL_{az}. \end{aligned} \qquad [A6]$$

For two droplets [A3] is valid and [A4] becomes:

$$\mathbf{F}_a^{(b)} = 6\pi\mu_e a \frac{2/3 + \lambda_a}{1 + \lambda_a} \boldsymbol{K}_b \cdot \mathbf{U}_b,$$

$$\mathbf{F}_b^{(a)} = 6\pi\mu_e b \frac{2/3 + \lambda_b}{1 + \lambda_b} \boldsymbol{K}_a \cdot \mathbf{U}_a,$$

which yields at last:

$$a\frac{2/3 + \lambda_a}{1 + \lambda_a} K_{bx} = b\frac{2/3 + \lambda_b}{1 + \lambda_b} L_{ax}; \qquad a\frac{2/3 + \lambda_a}{1 + \lambda_a} K_{bz} = b\frac{2/3 + \lambda_b}{1 + \lambda_b} L_{az}.$$